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The Regulator Theory for Finite Automata*

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This paper deals with the problem of controlling the state of a finite automaton to some preassigned value; namely, the regulator problem. First, the simplest situation where the state of the system is currently known is briefly considered and the corresponding solution pointed out. Then, the much more difficult problem arising in the case of nontrivial output transformation is analyzed. This obviously calls for the preliminary solution of the state reconstruction problem in a partially or totally uncertain environment.

The obtained results, as compared with the present status of the regulator theory for linear dynamical systems, allow us to point out strong similarities as well as definite differences, the latter being basically due to the essentially nonlinear nature of the systems considered herein.

1. INTRODUCTION

Given a dynamical system S, the regulator problem basically consists in controlling S in such a way that its state is kept, as far as possible, fixed at some suitably preassigned value and led back to (a neighborhood of) it whenever the system behavior is upset by an accidental perturbation.

From the very beginning of control theory (Maxwell, 1866), this problem has received a great deal of attention even if, despite of its deceiving simplicity, only in the last few years and only in the simplest case, namely in the case of linear difference or differential systems, it can be considered satisfactorily solved (Kalman, Falb, and Arbib, 1969). The systems dealt with in this paper are finite automata. The importance of a regulator theory for this class of systems lies in two kinds of considerations: From a practical point of view, an increasing number of controlled systems, mainly in the economic and biologic area, can be conveniently described through a finite automaton, while, from a conceptual point of view, strong and somewhat unexpected similarities (as well as differences) with the linear difference or differential case can be pointed

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out by contemporarily getting a deeper insight into the general nonlinear problem.

As customary, a key role in determining the structure of the problem solution (open versus closed loop control) is played by the notion of uncertainty. A purely deterministic description of the uncertainty will be adopted throughout this paper, in order to establish the main results in the simplest mathematical form. Actually, a stochastic description of uncertainty, although very useful in a huge number of problems, is not at all necessary to point out the few relevant aspects which do basically affect the regulator problem.

Specifically, the control system will be assumed to be possibly perturbed by a disturbance, acting over a finite interval of time; correspondingly, the regulator will be required to lead back the state of the controlled system to the desired value and to keep it there no matter of what kind of modifications the disturbance has produced during its action. Of course, the above assumption and the corresponding problem specification can be considered the fairly idealized description of a more realistic situation in which the occurrence of a disturbing action is generally followed by a sufficiently long interval of time during which no disturbance does actually affect the controlled system.

As it was conceivable to expect, it will be shown that, whenever the state of the controlled system can be assumed to be currently known (measurable), then disturbances of the above mentioned kind do not affect at all the feedback regulator design. A different situation occurs whenever a nontrivial output–transformation makes the state of the system to be possibly uncertain. In fact, the presence or absence of any disturbance affecting the behavior of the controlled system must be taken into account in order to make a proper design of the state reconstructor.

The present paper is organized in sections as follows.

After some preliminary definitions and results (Section 2), the state-feedback regulator problem is dealt with and solved in Section 3. Next, Section 4 deals with the crucial problem of the state reconstructor design. Here, a distinction between synchronous and asynchronous reconstruction is introduced. The asynchronous reconstruction problem is said to possess a solution if an exact reconstruction of the state can be achieved in a finite time and in a totally uncertain situation, i.e., without any information on whether disturbances are actually affecting the system or not; whenever, on the contrary, the reconstructor can be assumed to be currently aware of the occurrence of some perturbing action (partial uncertainty) in such a way that, after each perturbation, its state can be reset to a suitably preassigned value, then the problem to solve is said to be one of synchronous reconstruction. In order to avoid any possible confusion, it may be worth noticing that the notions of synchronous and asynchronous reconstruction, as used in the present paper, have nothing to do with the definitions of synchronous or asynchronous machines usually introduced in switching theory (see, for instance, Miller (1965), Vol. II, Chapter 9). Finally, the results obtained in Section 4 are used in Section 5 to completely solve the main problem dealt with in this paper, namely the output-feedback regulator problem, while a few comments and concluding remarks are given in Section 6.

From a more general point of view, this paper may be seen as a further contribution to bridging the gap between automata and control theory, along the line started by Arbib (1966), continued by Massey and Sain (1967), and then spread in many directions (see, for instance, Kambayashi and Yajima, 1972).

2. Preliminaries

Let the system under control be a finite automaton S described by

$$x(t+1) = f(x(t), u(t)), \tag{1a}$$

$$y(t) = g(x(t), u(t)), \tag{1b}$$

where, $\forall t \in Z$ (the set of integers),

$$u(t) \in U = \{u_1, u_2, ..., u_m\}, \tag{1c}$$

$$x(t) \in X = \{x_1, x_2, ..., x_n\}, \tag{1d}$$

$$y(t) \in Y = \{y_1, y_2, ..., y_p\}. \tag{1e}$$

$f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ are given functions from $X \times U$ into X and Y, respectively. The set Ω of admissible input functions is assumed to be the free semigroup U^+ generated by U. Furthermore, for any $t, \tau \in Z$, $t > \tau$, $x \in X$, $u_{[\tau,t)}(\cdot) \in U^+$, let $\phi(t; \tau, x, u_{[\tau,t)}(\cdot))$ be state of S at time t, once the sequence $u_{[\tau,t)}(\cdot)$ has been applied, starting at time τ from state x.

Definition 1. A state $x_i \in X$ is controllable to $x_j \in X$ if there exist $t \in Z$ and $u_{[0,t)}(\cdot) \in U^+$ such that $\phi(t; 0, x_i, u_{[0,t)}(\cdot)) = x_j$.

Definition 2. The automaton S is controllable to a state $x_j \in X$ if every state of S is controllable to x_j. ▮

So far, let $G = (X, A)$, $A \subset X \times X$, be the oriented graph associated with

S (for a formal definition, see Starke (1972), p. 25). A path of G is a sequence of elements of A (namely of arcs): $[(x_0, x_1), (x_1, x_2),..., (x_{k-1}, x_k)]$.

A path is termed elementary if all nodes appearing along the path are distinct. A cycle is a path for which $x_0 = x_k$. A self-loop is a cycle with $k = 1$. If (x_i, x_j) or (x_j, x_i) belongs to A, both (x_i, x_j) and (x_j, x_i) are edges of G. A sequence of edges $[(x_0, x_1), (x_1, x_2),..., (x_{k-1}, x_k)]$ is said to form a chain. A chain is closed if $x_0 = x_k$. If every pair of distinct nodes of G are joined by a chain, then G is connected. A tree is a connected graph which contains no closed chains. A spanning tree of G is a tree with $n - 1$ arcs. If there is a path from x_i to x_j, for all $j \neq i$, then x_i is a root. The graph obtained by reversing all the arcs of A is the inverse of G. Then, the following simple result (whose proof is omitted as cumbersome and conceptually almost trivial) can be stated.

THEOREM 1. *The automaton S is controllable to $x_j \in X$ iff there exists in G an inverse spanning tree rooted in x_j.*

The remainder of this section is devoted to the introduction of a stability concept which is strictly analogous to the global stability (in the Liapunov sense) of dynamical systems whose state set is a Euclidean space. With this aim, the definitions of free automaton and of equilibrium state are first recalled.

DEFINITION 3. A free automaton S_f is one the motion of which cannot be influenced by any control action (Ω is a singleton); it can be therefore described by

$$x(t + 1) = F(x(t)), \tag{2a}$$

$$y(t) = G(x(t)), \tag{2b}$$

where $F(\cdot)$ and $G(\cdot)$ are functions from X into X and Y, respectively.

DEFINITION 4. A state $\bar{x} \in X$ is an equilibrium state of S if there exists $\bar{u} \in U$ such that $f(\bar{x}, \bar{u}) = \bar{x}$.

Obviously, $\bar{x}$ is an equilibrium state of S_f if $F(\bar{x}) = \bar{x}$. ▮

Finally, for any $t, \tau \in Z$, $t > \tau$, $x \in X$, let $\Phi(t; \tau, x)$ be the state of S_f at time t, starting at time τ from state x. Then, the stability of an equilibrium state of S_f can be defined as follows.

DEFINITION 5. An equilibrium state $\bar{x}$ of S_f is stable if there exists $\bar{t} > 0$ such that $\Phi(t; 0, x) = \bar{x}$, for all $t \geqslant \bar{t}$ and $x \in X$. ▮

Notice that, if $\bar{x}$ is stable then a $\bar{t} < n$ can always be found such that $\Phi(t; 0, x) = \bar{x}$, for all $t \geq \bar{t}$ and $x \in X$. Furthermore, if an equilibrium state of S_f is stable, then it is unique.

The following theorem, given without proof, is a straightforward consequence of Definition 5.

THEOREM 2. *An equilibrium state $\bar{x}$ of S_f is stable iff the graph G_f associated with S_f is the union of a self-loop in $\bar{x}$ with an inverse spanning tree rooted in $\bar{x}$.*

3. THE STATE-FEEDBACK REGULATOR

Let $\bar{x}$ be an equilibrium state of S. The problem dealt with in this section is the synthesis of a feedback control law $k(\cdot)\colon X \to U$ such that $\bar{x}$ is the unique stable equilibrium state of the free automaton S_f (Fig. 1) obtained from S by setting $u(t) = k(x(t))$.

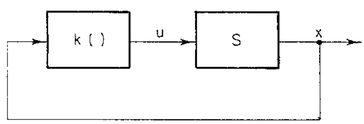

FIGURE 1

The solution to this problem is given by the following theorem.

THEOREM 3. *A feedback control law $k(\cdot)\colon X \to U$ such that $\bar{x}$ is a stable equilibrium state of S_f exists iff S is controllable to $\bar{x}$.*

Proof. Necessity is a trivial consequence of Definitions 2 and 5. As for sufficiency, if S is controllable to an equilibrium state $\bar{x}$, then there exist in the graph G associated with S a self-loop in $\bar{x}$ and, in view of Theorem 1, an inverse spanning tree rooted in $\bar{x}$. They obviously define a control law $k(\cdot)\colon X \to U$ such that the graph G_f associated with S_f is just their union. Hence, in view of Theorem 2, $\bar{x}$ is a stable equilibrium state of S_f. ∎

The remainder of this section is devoted to pointing out an interesting property of S_f which, as a feedback control system, has the expected capability of tolerating accidental perturbations over a finite interval of time. The requirement that this capability be preserved when dealing with automata characterized by nontrivial output transformations (unaccessible state variable) will be shown to play an important role in the synthesis of the state reconstructor which is then an essential part of the feedback regulator (see Section 5).

Notice that whenever the dynamics and the state of the automaton S under control are perfectly known, then the nominal and the actual behavior of S do coincide; thus, given $x(0)$, it is possible to set up an open loop control system (Fig. 2) whose control action is just the same ($S_{\mathrm{nom}} = S$) as the one performed by S_f (Fig. 1). This is no longer true if the state variable of S is affected by accidental perturbations; in fact, while it is easy to see that the state of S_f will reach in any case $\bar{x}$ after a finite number of steps, this might not be the case for the system of Fig. 2, as the following elementary example shows.

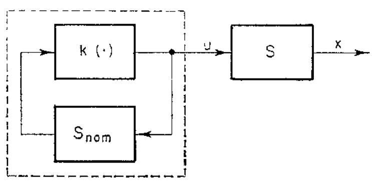

FIGURE 2

EXAMPLE. Let $U = \{0, 1\}$, $X = \{a, b, c\}$ and $f(\cdot, \cdot)$ be defined as follows:

$$f(a, 0) = a, \qquad f(b, 0) = c, \qquad f(c, 0) = a,$$

$$f(a, 1) = b, \qquad f(b, 1) = a, \qquad f(c, 1) = c.$$

Furthermore, let $\bar{x} = c$ be the equilibrium state of S to be stabilized. In this case, it is immediate to see that the feedback control law

$$k(\cdot)\colon k(a) = k(c) = 1, \qquad k(b) = 0$$

solves the problem. As for the implementation, while the system of Fig. 1 does obviously tolerate any upset of the state of S, in the sense that $\bar{x}$ will be reached anyway after a finite number of steps ($\leqslant 2$), it is easy to check that any upset of the state of S in the system of Fig. 2 leads the system itself to permanently switch between a and b, a completely undesired behavior. ∎

Finally it should be apparent that, as far as the above property is concerned, any perturbation of the dynamics of S over a finite interval of time is strictly equivalent to an upset of the state.

4. Synchronous and Asynchronous Reconstruction

This section deals with the well-known problem of state reconstruction. This problem arises whenever the unavailability of the state of S makes it impossible to stabilize $\bar{x}$ via a state feedback regulator. Actually, a basic part of the output feedback regulator will be shown to consist of a reconstructor; namely, of a finite automaton R whose inputs are the input and the output of S and whose output is, at each time, a subset of X which is expected to include the current state of S. When such a subset consists of a single element, the state of S is said to be completely reconstructed.

Let $\hat{U} = U \times Y$ and $\hat{X} \subset P(X)$, where $P(\cdot)$ is the power set, be the input and state sets of R, respectively.

Definition 6. The automaton R is a solution of the open loop asynchronous reconstruction problem if $\hat{X} \supset \{X\}$ and there exist $u(\cdot)$ and $T \in Z$ such that

$$\hat{x}(t) = \{x(t)\}, \qquad \forall t \geqslant T,$$

whatever $\hat{x}(0) \in \hat{X}$ and $x(0) \in X$ may be.

If, on the contrary, the above properties hold only for all $\hat{x}(0) \in \hat{X}$ and $x(0) \in \hat{x}(0)$, then R is said to be a solution of the open loop synchronous reconstruction problem.

Conversely, R is a solution of the closed loop asynchronous (synchronous) reconstruction problem if $\hat{X} \supset \{X\}$ and there exist a feedback control law $\hat{k}(\cdot): \hat{X} \to U$ and $T \in Z$ such that

$$\hat{x}(t) = \{x(t)\}, \qquad \forall t \geqslant T,$$

for all $\hat{x}(0) \in \hat{X}$, $x(0) \in X$ ($x(0) \in \hat{x}(0)$). ∎

Notice that if R is a solution of an asynchronous reconstruction problem, then it is obviously also a solution of the corresponding synchronous problem.

A more important concern is with the initial correctness condition ($x(0) \in \hat{x}(0)$) appearing in Definition 6. Such a condition, on which the distinction between synchronous and asynchronous reconstruction is based, can obviously be met with by setting $\hat{x}(0) = X$.

However, it will be shown in the sequel that the concept of asynchronous reconstruction plays an essential role when systems possibly affected by completely unknown accidental perturbations must be considered. As a matter of fact, this situation, henceforth referred to as one of total uncertainty, apparently calls for an asynchronous reconstruction as it can easily be seen by noting that only if the occurrence of possible perturbations is somehow perceived, i.e., if a binary (activity-inactivity) information concerning the disturbance can be assumed to be available (partial uncertainty), it is really possible, whenever a disturbance occurs, to reset the state of R to X, so that nothing more than a synchronous reconstruction is finally required; in all other cases, any useful information about disturbances must be derived from input-output data and, by definition, this is a job that any solution of an asynchronous reconstruction problem shall be able to perform.

The state-reconstruction problem, often discussed under the heading of Measurement and Identification, is obviously a fundamental and self-motivated problem that has been widely explored in automata theory; however, the available results basically do refer to unperturbed systems in reduced (or minimal) form. These classic results can easily be adjusted in order to obtain a solution of the synchronous reconstruction problem by contemporarily introducing a slight modification thanks to which the minimality assumption can be relaxed. With this aim, a particular reconstructor, henceforth referred to as basic reconstructor, is now introduced, and a few facts are recalled which are almost trivial consequences of results already available in the literature (see, for instance, Gill (1962), Chapter 4, or Starke (1972), Part I, Section 9). The most important of these facts is stated without proof as Theorem 4.

First of all, for any function $\gamma(\cdot): \Delta \to \Gamma$ and any $D \subset \Delta$, define

$$\gamma(D) \triangleq \{w : w = \gamma(d), d \in D\}.$$

Then, for any $y \in g(\hat{x}, u)$, let

$$B(\hat{x}, u, y) \triangleq \{x : x \in \hat{x}, g(x, u) = y\}$$

and

$$A(\hat{x}, u) \triangleq \{\hat{x}' : \hat{x}' = f(B(\hat{x}, u, y), u), y \in g(\hat{x}, u)\}.$$

Furthermore, consider the following recursion

$$\hat{X}^{j+1} = \hat{X}^j \cup Z^j$$

where

$$Z^j \triangleq \bigcup_{\hat{x} \in \hat{X}^j} \bigcup_{u \in U} A(\hat{x}, u) \qquad \text{and} \qquad \hat{X}^1 \triangleq \{X\}.$$

It is straightforward to see that

(i) $\hat{X}^j \subset \hat{X}^{j+1} \subset \cdots \subset P(X)$, $\forall j$,

(ii) $\hat{X}^j = \hat{X}^{j+1} \Rightarrow \hat{X}^j = \hat{X}^{j+i}$, $\forall i > 1$.

Let μ be the smallest value of j such that $\hat{X}^j = \hat{X}^{j+1}$; then $\hat{X}^\mu$ is the state set of the basic reconstructor, whose next-state function

$$f_R(\cdot, \cdot): (\hat{x}, (u, y)) \mapsto f(B(\hat{x}, u, y), u)$$

is defined for all $\hat{x} \in \hat{X}^\mu$, $u \in U$ and $y \in g(\hat{x}, u)$.

Recall now that two states $x_1, x_2 \in X$ are said to be equivalent if

$$g(\phi(t; 0, x_1, u_{[0,t)}(\cdot)), u(t)) = g(\phi(t; 0, x_2, u_{[0,t)}(\cdot)), u(t)),$$

for all $t \geqslant 0$ and all $u(\cdot) \in U^+$. Furthermore, notice that if, for each equivalence class $\mathscr{X} \subset X$ of S, there exist $T \in Z$ and $u(\cdot)$ such that

$$\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot)), \qquad \text{for all} \quad x', x'' \in \mathscr{X},$$

then the set

$$\tilde{X} \triangleq \{\hat{x}: \hat{x} \in \hat{X}^\mu, |\hat{x}| = 1\}$$

is nonempty. On the other hand, it is obvious that $\hat{X}^\mu \supset \{X\}$.

THEOREM 4. *The open loop synchronous reconstruction problem for system S admits a solution R iff, for each equivalence class $\mathscr{X} \subset X$ of S, there exist $T \in Z$ and $u(\cdot)$ such that*

$$\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot)), \qquad \textit{for all} \quad x', x'' \in \mathscr{X}.$$

Such a result is usually proved under the slightly simplifying assumption that S is in minimal form, i.e., that each equivalence class induced on X is a singleton, however, the extension to the more general case is almost trivial.

It goes without saying that a solution (if any) of the open loop synchronous reconstruction problem is the basic reconstructor defined above. The same holds for the closed loop synchronous reconstruction problem, in view of the following result.

THEOREM 5. *The closed loop and the open loop synchronous reconstruction problems are equivalent*

Proof. First, assume that R is a solution of the open loop synchronous reconstruction problem (say, the basic reconstructor). Then, for each $\hat{x}(0) \in \hat{X} - \tilde{X}$, there must exist at least one control $u^0(\cdot; \hat{x}(0))$ such that the maximum reconstruction time (with respect to all $x(0) \in \hat{x}(0)$) is minimum; let $\tau(\hat{x}(0))$ be such a minimum reconstruction time.

Hence, define

$$\hat{k}(\hat{x}(0)) \triangleq u^0(0; \hat{x}(0)), \qquad \forall \hat{x}(0) \in \hat{X} - \tilde{X}$$
$$\triangleq k(x(0)), \qquad \forall \hat{x}(0) \in \tilde{X}$$

where $x(0) \in \hat{x}(0)$ and $k(\cdot)$ is an arbitrary function from X into U. All functions $\hat{k}(\cdot)$ will henceforth be referred to as MMRT (Min Max Reconstruction Time) feedback control laws. System R can now be shown to constitute also a solution of the closed loop synchronous reconstruction problem. In fact, for each $\hat{x}(0) \in \hat{X} - \tilde{X}$, let $M(\hat{x}(0))$ be the subset of $\hat{X}$ whose elements are reached from $\hat{x}(0)$ when $u(\cdot)$ is fixed to $u^0(\cdot; \hat{x}(0))$ and $x(0)$ ranges over $\hat{x}(0)$. Then

$$\tau(\hat{x}) < \tau(\hat{x}(0)), \qquad \forall \hat{x} \in M(\hat{x}(0)),$$

so that

$$\hat{x} \in M(\hat{x}(0)) \Rightarrow \hat{x}(0) \notin M(\hat{x});$$

therefore, each MMRT feedback control law $\hat{k}(\cdot)$ forces the state of R along trajectories (or paths) which must be elementary over $\hat{X} - \tilde{X}$ and in view of the finiteness of $\hat{X}$, there must exist $T \in Z$ such that

$$\hat{x}(t) \in \tilde{X}, \qquad \forall t \geqslant T.$$

Conversely, assume that the closed loop synchronous reconstruction problem has a solution.

Then, for each equivalence class $\mathscr{X} \subset X$, apparently there must exist $T \in Z$ and $u(\cdot)$ such that $\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot))$, $\forall x', x'' \in \mathscr{X}$. Hence, in view of Theorem 4, the open loop synchronous reconstruction problem also admits a solution. ▌

Finally, the closed loop asynchronous reconstruction problem, which is the main problem involved by the feedback regulator theory, must be considered. Its solution is given by Theorem 6 below, the statement of which requires some further preliminary definitions.

DEFINITION 6. A pair of periodic functions $(u(\cdot), y(\cdot))$ is a fundamental periodic regime of S if there exists a unique $x \in X$ such that $\phi(\cdot; 0, x, u(\cdot))$ is periodic too and

$$g(\phi(t; 0, x, u(\cdot)), u(t)) = y(t), \qquad \forall t \geqslant 0.$$

DEFINITION 7. A fundamental cycle is the unique closed state-trajectory corresponding to a fundamental regime.

DEFINITION 8. A state $x_i \in X$ is controllable to a cycle $[(x_0, x_1), (x_1, x_2), \ldots, (x_{k-1}, x_0)]$ if it is controllable to some $x_j \in \{x_0, x_1, x_2, \ldots, x_{k-1}\}$.

THEOREM 6. *The closed loop asynchronous reconstruction problem has a solution iff*

(i) *for each equivalence class $\mathscr{X} \subset X$, there exist $T \in Z$ and $u(\cdot)$ such that*

$$\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot)), \qquad \forall x', \quad x'' \in \mathscr{X};$$

(ii) *every state of X is controllable to at least one fundamental cycle.*

Proof. Necessity. This part of the proof is by contradiction. If condition (i) is violated, then, in view of Theorems 4 and 5, the closed loop synchronous reconstruction problem has no solution; then neither has one the closed loop asynchronous reconstruction problem. On the other hand, consider the system of Fig. 3, where R is a solution of the closed loop asynchronous reconstruction problem, and assume condition (ii) is violated. Then, there exists $x^* \in X$ such that, for any arbitrary $\hat{x} \in \hat{X}$, the free automaton of Fig. 3, with initial state $(x^*, \hat{x})$, reaches in a finite time a cycle, corresponding to which the pair $(u(\cdot), y(\cdot))$ is not a fundamental periodic regime of S. Hence, letting $(x_0, \hat{x}_0)$ be a state of such a cycle and $(u_0(\cdot), y_0(\cdot))$ be the corresponding nonfundamental periodic regime of S, there must exist $x_0' \neq x_0$ such that $\phi(\cdot; 0, x_0', u_0(\cdot))$ is a periodic function and

$$g(\phi(t; 0, x_0', u_0(\cdot)), u_0(t)) = y_0(t), \qquad \forall t \geqslant 0.$$

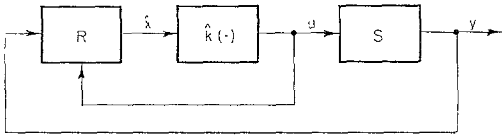

FIGURE 3

Therefore, the initial state of the free automaton being either $(x_0, \hat{x}_0)$ or $(x_0', \hat{x}_0)$, the corresponding periodic motions of R must coincide, while the motions of S do apparently differ. This contradicts the assumption that R is a solution of the closed loop asynchronous reconstruction problem.

Sufficiency. Recall that up to now the next state function of the basic reconstructor has been defined on a subset of $U \times Y$ only. However, it can easily be completely specified as follows:

$$\begin{aligned} f_R(\hat{x}, (u, y)) &\triangleq f(B(\hat{x}, u, y), u), \qquad && y \in g(\hat{x}, u), \\ &\triangleq X, && y \notin g(\hat{x}, u), \end{aligned}$$

where

$$B(\hat{x}, u, y) \triangleq \{x \colon x \in \hat{x}, g(x, u) = y\}.$$

The so obtained automaton R is now proved to be a solution of the closed loop asynchronous reconstruction problem. In fact, let $\hat{k}(\cdot)$ be a MMRT feedback control law (see proof of Theorem 5) whose restriction to $\tilde{X}$, when taken as state-feedback control law of S, be such that each state of X is led (in a finite number of steps) to a fundamental cycle of S. Following standard arguments it is easy to prove that, in view of conditions (i) and (ii), such a function does always exist. It has then to be shown that, corresponding to each periodic motion of the free automaton $\hat{S}_f$ of Fig. 3, where R and $\hat{k}(\cdot)$ are defined as specified above, it must be $\hat{x}(t) = \{x(t)\}$, $\forall t$. This part of the proof is in three steps. First, the closed trajectory described by the state of R is all contained in $\tilde{X}$. In fact, this is obviously true whenever $x(t) \in \hat{x}(t)$, for some t, since in that case a synchronous reconstruction occurs; on the other hand, even if $x(t) \notin \hat{x}(t)$, $\forall t$, it must be $y(t) \in g(\hat{x}(t), u(t))$, $\forall t$, since $\hat{x}(t) \neq X$; hence, through the same argument used in the synchronous case, the conclusion can be drawn that $\hat{k}(\cdot)$ forces the state of R along trajectories which must be elementary over $\hat{X} - \tilde{X}$. As a second step, let $\xi(\cdot)$ be the periodic motion of S such that $\hat{x}(t) = \{\xi(t)\}$, $\forall t$, then the closed trajectory in X corresponding to $\xi(\cdot)$, is a fundamental cycle of S. In fact, $y(t) \in g(\hat{x}(t), u(t))$, i.e., $y(t) = g(\xi(t), u(t))$, $\forall t$, implies that $B(\{\xi(t)\}, u(t), y(t)) = \{\xi(t)\}$; hence, letting $\tilde{k}(\cdot)$ be the restriction of $\hat{k}(\cdot)$ to $\tilde{X}$,

$$\{\xi(t+1)\} = f_R(\{\xi(t)\}, (u(t), y(t))) = f(\{\xi(t)\}, u(t)) = f(\{\xi(t)\}, \tilde{k}(\xi(t)))$$

which implies

$$\xi(t+1) = f(\xi(t), \tilde{k}(\xi(t))).$$

Since, by definition, $\tilde{k}(\cdot)$ is such that each state of X is led to a fundamental cycle of S, the periodic motion $\xi(\cdot)$ must correspond to a fundamental cycle of S. The final step is the following. Since $(u(\cdot), y(\cdot))$ is a fundamental periodic regime of S, the actual motion of S, i.e., $x(\cdot)$, must coincide with $\xi(\cdot)$ (see Definition 7).

5. The Output-Feedback Regulator

The problem dealt with in this section can be stated in the following way. Let $\bar{x}$ be an equilibrium state of S. Then, referring to Fig. 4, find a dynamical system C, with input y and output u, such that the state of S is led in a finite time to $\bar{x}$, whatever its initial value may be.

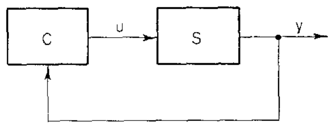

Figure 4

As for the information structure, i.e. the information the controller is supposed to be aware of about the disturbance activity, it has already been pointed out in the Introduction and in Section 4 that two situations are of interest. The first one (total uncertainty) occurs when no information about the disturbance comes to C but through y. The second situation (partial uncertainty) refers to the case where a current information on whether some disturbance is acting on S or not is anyhow supposed to reach C. Since any disturbance acting on a finite interval of time can be represented as an upset of the initial state of S, it must be consistently assumed that the state of C can be reset, whenever a disturbance occurs, to a suitably preassigned value, only when a partially uncertain situation must be faced. On the other hand, it should be apparent that in a totally uncertain situation the output-feedback regulator C must lead back the state of S from any initial value in X to the desired one (and keep it there) regardless of what the value of its own initial state may be.

Necessary and sufficient conditions for the output-feedback regulator problem to have a solution are given by Theorem 7, for the case of partial uncertainty, and by Theorem 8 for the totally uncertain situation. In both cases, it will be shown that a finite-state regulator C can easily be designed as a tandem connection of a state-reconstructor and an algebraic control law, in deep agreement with the results achieved on the regulator problem in linear system theory (Kalman, Falb, and Arbib, 1969).

Theorem 7. *In a partially uncertain situation, the output-feedback regulator problem has a solution iff*

(i) *S is controllable to $\bar{x}$,*

(ii) *for each equivalence class* $\mathscr{X} \subset X$, *there exist* $T \in Z$ *and* $u(\cdot)$ *such that*

$$\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot)), \qquad \textit{for all} \quad x', x'' \in \mathscr{X}.$$

Proof. Necessity. Condition (i) is obvious. As for condition (ii), let x', x'' be two states of $\mathscr{X}$. Then, there exist, by assumption, a controller C (see Fig. 4) and an initial state of C such that the state of S is led, in a finite time, to $\bar{x}$ no matter whether its initial value is x' or x''. Since x' and x'' belong to the same equivalence class, the control action $u(\cdot)$ performed by C cannot depend on whether the initial state of S is x' or x''; hence, (ii) holds.

Sufficiency. In view of condition (i) and Theorem 3, there exists a solution $\bar{k}(\cdot)$ of the state-feedback regulator problem. On the other hand, by virtue of condition (ii) and Theorems 4 and 5, the closed loop synchronous reconstruction problem admits a solution; hence, the system of Fig. 3, where R is the basic reconstructor (with $\hat{x}(0) = X$) and $\hat{k}(\cdot)$ is the MMRT feedback control law (see proof of Theorem 5) whose restriction to $\tilde{X}$ is equal to $\bar{k}(\cdot)$, apparently solves the output-feedback regulator problem.

THEOREM 8. *In a totally uncertain situation, the output-feedback regulator problem has a solution iff*

(i) *S is controllable to $\bar{x}$,*

(ii) *the self-loop associated to $\bar{x}$ is a fundamental cycle.*

Proof. Necessity. Again, condition (i) is obvious. As for condition (ii), consider Fig. 4 and let $(\bar{u}(\cdot), \bar{y}(\cdot))$ be a periodic regime of S corresponding to which $x(t) = \bar{x}$, $\forall t$. Since the output-feedback regulator problem admits, by assumption, a solution in the totally uncertain case, it follows that (a) the motion of S corresponding to any possible periodic regime of the control system of Fig. 4 must be such that $x(t) = \bar{x}$, $\forall t$; (b) the pair $(\bar{y}(\cdot), \bar{u}(\cdot))$ is a periodic regime of C. Now assume, by contradiction, that the self-loop associated to $\bar{x}$ is not a fundamental cycle; then, the system of Fig. 4 should admit a periodic regime corresponding to which $u(\cdot) = \bar{u}(\cdot)$, but $x(t) \neq \bar{x}$, for some t. Hence, (ii) holds.

Sufficiency. First of all, it will be proved that, from conditions (i) and (ii), it follows that, for each equivalence class $\mathscr{X} \subset X$, there exist $T \in Z$ and $u(\cdot)$ such that $\phi(T; 0, x', u(\cdot)) = \phi(T; 0, x'', u(\cdot))$, $\forall x', x'' \in \mathscr{X}$. In fact, let n be the cardinality of X, $u'(\cdot)$ be the control function leading the state of S from x' to $\bar{x}$ (see condition (i)), $\bar{u} \in U$ and $\bar{y} \in Y$ be such that $f(\bar{x}, \bar{u}) = \bar{x}$ and

$g(\bar{x}, \bar{u}) = \bar{y}$. Consider the control function consisting of $u'(\cdot)$ followed by a sequence of $\bar{u}$ of length n. This control function leads S to a periodic regime corresponding to which $u(t) = \bar{u}$ and $y(t) = \bar{y}$, $\forall t$, whatever the initial state of S in $\mathscr{X}$ may be. In view of condition (ii), to such a periodic regime there must correspond the self-loop in $\bar{x}$, only.

Thus, conditions (i) and (ii) imply (Theorem 6) that the closed loop asynchronous reconstruction problem can actually be solved by a suitably designed finite automaton (Fig. 3). Such an automaton also solves the output-feedback regulator problem, provided that the restriction to $\tilde{X}$ of $\hat{k}(\cdot)$, which, according to the proof of Theorem 6, must be such that each state of X is led (in a finite interval of time) to a fundamental cycle of S, be an arbitrary solution of the state-feedback regulator problem. ∎

6. Concluding Remarks

In this paper, the bases of the regulator theory for finite automata have been established. In comparison with the now well-settled regulator theory as it has been developped for linear difference or differential systems, strong similarities as well as definite differences may be pointed out.

In fact, the concepts of controllability, stability, and reconstructability are, in any case, of fundamental importance, and it may be said that the main relationships among them do not depend upon the particular class of dynamical systems taken into consideration. Furthermore, one of the most interesting features of classic feedback regulators, namely a sort of long-term immunity against any kind of disturbances acting on a finite interval of time, can be preserved in the case of finite automata, regardless of whether the state of the system is directly accessible or not. As for the differences, first it may be pointed out that, in the case of finite automata with accessible state, the concept of controllability is equivalent to the one of feedback stabilizability, while this is not the case for linear difference or differential systems, where uncontrollability is allowed, provided that the uncontrollable part of the system is known to be stable. Secondly, it is worth noticing that an interesting property which can be met with when designing a linear difference or differential regulator, namely the long-term immunity against possible signal disturbances of polynomial type affecting the controlled system, seems not to make sense in the case of finite automata.

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